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Note on a duality of topological branes

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We show a duality of branes in the topological B-model by inserting two kinds of non-compact branes simultaneously. We explicitly derive the integral formula for the matrix model partition function describing this situation, which correspondingly includes both the characteristic polynomial and the external source. We show that these two descriptions are dual to each other through the Fourier transformation, and the brane partition function satisfies integrable equations in one and two dimensions.

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1. Introduction

The D-brane is the most fundamental, but also non-perturbative object in superstring theory. It gives a nontrivial boundary condition for open strings, and consequently a lot of gauge theories can be constructed as effective theories for stringy modes on them. When we implement various gauge groups and matter contents for the effective field theory, branes often have to be arranged in a complicated way. In such a case, we have to deal with the dynamics of brane intersection appropriately, and discuss what kind of stringy excitation can be allowed there. In this sense it is important to study the properties of a brane complex in detail.

In this article we investigate some aspects of brane-on-brane structure in topological string theory, especially through its matrix model description, by introducing two kinds of branes simultaneously. It is well known that the Riemann surface plays a crucial role in the topological B-model on Calabi–Yau threefolds, and this Riemann surface can be identified with the spectral curve appearing in the large N limit of the matrix model [1]. A brane is also introduced into the B-model, and its realization in terms of the matrix model has been extensively investigated [2]: There are seemingly two kinds of non-compact branes, which correspond to the characteristic polynomial and the external source in the matrix model. While these two descriptions are apparently different, we will see that they are essentially the same, as a consequence of the symplectic invariance of the topological B-model. From the viewpoint of the matrix model, this situation just corresponds to insertion of both the characteristic polynomial and the external source. We will explicitly show that two descriptions are dual to each other by seeing a newly derived matrix integral formula.

It is known that the correlation function of the characteristic polynomial and the matrix integral with the external source are deeply related to the integrable hierarchy [3]. Thus it is naively expected that the hybrid of them also possesses a similar connection to the integrability. Since in this case we have two kinds of variables corresponding to the characteristic polynomials and the external

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sources, we obtain the 2D integrable equation as a consequence, and identify the partition function as the τ-function of the corresponding integrable hierarchy with the Miwa coordinates.

This article is organized as follows. In Sect. 2 we start with a review on how to consider branes in the topological B-model in terms of the matrix model. We introduce two descriptions of branes using the characteristic polynomial and the external source of the matrix integral with emphasis on the similarities and differences between them. In Sect. 3 we show a duality between two kinds of non-compact branes in the B-model by introducing both of them at the same time. We derive a new matrix integral formula corresponding to this situation, and show that the characteristic polynomial and the external source are dual to each other in the sense of Fourier transformation. We also show that the brane partition function satisfies the integrable equation in one and two dimensions, which is well known as the Toda lattice equation. In Sect. 4 we consider the Gaussian matrix model as an example. In this case we can show the duality using the fermionic variables, which is useful to discuss the effective degrees of freedom on the branes. We close this article in Sect. 5 with some discussions and concluding remarks.

2. Branes in the topological B-model

We first review the large N matrix model description of the topological B-model, mainly following Ref. [2], and how to introduce the brane, which is also called the B-brane. The non-compact Calabi–Yau threefold that we discuss in this article is obtained as a hypersurface in $(u, v, p, x) \in \mathbb{C}^4$:

$$u\,v - H(p, x) = 0. \tag{1}$$

Here $H(p, x)$ determines the Riemann surface

$$\Sigma: \; H(p, x) = 0 \tag{2}$$

with

$$H(p, x) = p^2 - W'(x)^2 - f(x), \tag{3}$$

where $W(x)$ and $f(x)$ are polynomials of degree $n + 1$ and n, respectively. This smooth Calabi–Yau threefold is given by deformation of the singular one:

$$H(p, x) = p^2 - W'(x)^2. \tag{4}$$

The branes, which are compact, are wrapping n S^3 at critical points $W'(x) = 0$, and the sizes of the S^3 s are parametrized by $f(x)$, which describes the quantum correction around them.

The holomorphic (3, 0)-form Ω in the Calabi–Yau threefold (1) is chosen to be

$$\Omega = \frac{du \wedge dp \wedge dx}{u}. \tag{5}$$

Then the periods of this (3, 0)-form Ω over three-cycles reduce to the integral of the symplectic 2-form

$$\int_D dp \wedge dx, \tag{6}$$

where D stands for domains in the complex 2D (p, x)-plane. The boundary of this domain denoted by γ is related to the Riemann surface (2) as $\gamma = \partial D \subset \Sigma$. Thus this integral further reduces to a one-cycle on the Riemann surface Σ:

$$\int_\gamma p\,dx. \tag{7}$$

In this way we can focus only on the complex 1D subspace Σ by keeping the dependence on u and v fixed.

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2.1. Characteristic polynomial

The algebraic curve (2) can be identified with the spectral curve of the matrix model: It is just given by the loop equation in the large N limit of the matrix integral:

$$\mathcal{Z}_N = \int dX e^{-\frac{1}{g_s}\operatorname{Tr} W(X)}. \tag{8}$$

The polynomial $W(x)$ is the matrix potential. This is the reason why we can discuss the topological B-model using the matrix model. In this description the genus expansion with $1/N$ corresponds to the quantum correction, and the 't Hooft parameter $t = g_s N$ gives the size of S^3.

The other canonical variable in (3) is related to the resolvent of the matrix model:

$$p(x) = W'(x) - 2g_s \operatorname{Tr} \frac{1}{x - X}. \tag{9}$$

The saddle point equation of the matrix model is equivalent to the condition $p(x) = 0$. Since the one-form on the spectral curve is given by these two canonical variables,

$$\lambda = p(x)\, dx = d\phi, \tag{10}$$

we can naturally introduce the chiral boson $\phi(x)$ on the Riemann surface Σ, which is interpreted as the Kodaira–Spencer field describing deformation of the complex structure at infinity:

$$\phi(x) = W(x) - 2g_s \operatorname{Tr}\log(x - X). \tag{11}$$

Here this $\phi(x)$ also has the meaning of the effective potential for the matrix model, and thus the eigenvalues are degenerate at the critical point, $\partial\phi = p(x) = 0$.

The vertex operator, which creates a non-compact brane at a position x, is constructed by the standard bosonization scheme:

$$V(x) = e^{-\frac{1}{2g_s}\phi(x)} = e^{-\frac{1}{2g_s}W(x)} \det(x - X). \tag{12}$$

The prefactor $e^{-\frac{1}{2g_s}W(x)}$ corresponds to the classical part of the operator, while the determinant part, namely the characteristic polynomial, gives the quantum fluctuation as a gravitational back reaction. This brane creation operator gives a pole at x on the Riemann surface, and its residue is given by

$$\oint \lambda = g_s. \tag{13}$$

This is just interpreted as one brane contribution, and also shows that g_s plays the role of the Planck constant $\hbar$ for the canonical pair (p, x).

The partition function of the branes is represented as a correlation function of the characteristic polynomials, e.g., the k-point function given by

$$\left\langle \prod_{\alpha=1}^{k} \det(x_\alpha - X) \right\rangle. \tag{14}$$

This expectation value is taken with respect to the matrix measure

$$\langle \mathcal{O}(X) \rangle = \frac{1}{\mathcal{Z}_N} \int dX\, \mathcal{O}(X)\, e^{-\frac{1}{g_s}\operatorname{Tr} W(X)} \tag{15}$$

with the standard normalization $\langle 1 \rangle = 1$. Including the classical part, the brane partition function is then given by

$$\Psi_k(x_1, \ldots, x_k) = \prod_{\alpha=1}^{k} e^{-\frac{1}{2g_s}W(x_\alpha)} \left\langle \prod_{\alpha=1}^{k} \det(x_\alpha - X) \right\rangle. \tag{16}$$

This correlation function can be exactly evaluated using the orthogonal polynomial method. We will come back to this formula in Sect. 3.2 (see (30)).

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2.2. External source

Let us then consider another kind of non-compact brane in the B-model, which is described by the external source in the matrix model. We consider the matrix action written in the form of

$$S(P) = \frac{1}{g_s} \mathrm{Tr}[W(P) - AP], \tag{17}$$

where the potential is regarded as an integral of the one-form along an open path to a certain point p on the Riemann surface:

$$W(p) = \int^{p} x(p')\, dp'. \tag{18}$$

We can assume that the matrix A is diagonal $A = \mathrm{diag}(a_1, \ldots, a_N)$ without loss of generality. The action (17) corresponds to the matrix model with the external source

$$\mathcal{Z}_N(A) = \int dP\, e^{-\frac{1}{g_s} \mathrm{Tr}[W(P) - AP]}, \tag{19}$$

which is analogous to the Kontsevich model [4]. The external source implies the positions of N branes, at least at the classical level, because the extremum of the action $W'(P) - A = 0$ gives the classical solution, $X = \mathrm{diag}(a_1, \ldots, a_N)$, since we have $W'(P) = X$ according to (18).

The one-form used in (18) is apparently different from (10), but they are equivalent up to the symplectic invariance for a pair of the canonical variables (p, x). This symmetry is manifest by construction of the topological B-model as seen in (5) and (6). Therefore, the two descriptions of the non-compact branes based on the characteristic polynomial and the external source in the matrix model are dual to each other in this sense. We will show in Sect. 3 that they are converted through the Fourier transformation by deriving the explicit matrix integral representation.

3. A duality of branes

As seen in the previous section, there are two kinds of non-compact branes in the topological B-model, which are related through the symplectic transformation. We now study the situation such that both kinds of branes are applied at once. This is realized by inserting the characteristic polynomial to the matrix model in addition to the external source. Let us consider the corresponding partition function denoted by

$$\Psi_{N,M}\left(\{a_j\}_{j=1}^{N}; \{\lambda_\alpha\}_{\alpha=1}^{M}\right) = \int dX\, e^{-\frac{1}{g_s} \mathrm{Tr}\, W(X) + \mathrm{Tr}\, AX} \prod_{\alpha=1}^{M} \det(\lambda_\alpha - X). \tag{20}$$

We now have to be careful of the meaning of the matrix potential and the external source. As discussed in Sect. 2.2, if the matrix potential is given by the integral of the one-form in the form of (18), the external source gives the classical positions of branes in the x-coordinate. In this case, however, the roles of x and p are exchanged. Instead of (18), the potential should be written as

$$W(x) = \int^{x} p(x)\, dx. \tag{21}$$

With this choice, the corresponding external source determines the positions of branes in the p-coordinate. We note that, in this case, the extremum of the action does not simply imply the classical positions of branes as $W'(X) = A$, because of the potential shift due to the characteristic polynomial. Although there exists this kind of back reaction, the interpretation of the external source as the p-coordinate still holds, as shown in the following.

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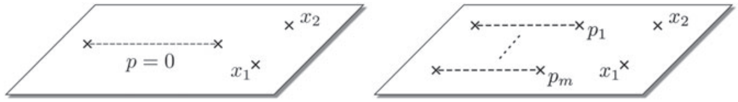

Fig. 1. (Left) In the case with $A = 0$, the one-cut matrix model has a single cut in the complex x-plane, which is seen as a pole at $p = 0$ with degree N in the p-plane. There are also simple poles created by the characteristic polynomials. (Right) When one applies the external source $A = \mathrm{diag}(p_1, \ldots, p_1, \ldots, p_m, \ldots, p_m)$, it is split into m distinct poles with the corresponding multiplicity.

The characteristic polynomial and the external source put punctures on the Riemann surface Σ for x and p coordinates because they correspond to the vertex operators creating the non-compact branes. Since (p, x) is a pair of the canonical variables, if once its x-coordinate is fixed, one cannot determine the other p-coordinate, and vice versa. This means that the positions of the branes created by the characteristic polynomial and the external source are fixed by the x and p coordinates. For example, in the case with $A = 0$ for simplicity, all the N eigenvalues are distributed on the extended line, namely the cut in the complex x-plane described by $p(x) = 0$, which corresponds to the one-form $p(x)dx$. On the other hand, if we apply the other representation based on the p-plane, the one-form $x(p)dp$ has a pole at $p = 0$ with degree N. Then, when one turns on the external source $A = \mathrm{diag}(a_1, \ldots, a_N)$, the pole at $p = 0$ is split and located at $p = a_i$ for $i = 1, \ldots, N$. In this way the external source and also the characteristic polynomial characterize a pole with the p and x coordinates, respectively. In Fig. 1 we illustrate the situation such that both kinds of branes are inserted to the Riemann surface Σ. We will see in the following that these coordinates can be exchanged through the Fourier transformation.

3.1. *Matrix integral formula*

We now provide an explicit formula for the partition function (20). In order to derive the formula, we apply a method to compute the matrix model partition function with the external source [5,6], which is also applicable to this situation.

First of all, we move to "eigenvalue representation" from the $N \times N$ Hermitian matrix integral (20) by integrating out the angular part of X. This can be done by using the Harish-Chandra–Itzykson–Zuber formula [7,8]

$$\int dU\, e^{\mathrm{Tr}\, UXU^\dagger Y} = \frac{\det e^{x_i y_j}}{\Delta(x)\Delta(y)}, \tag{22}$$

where the integral is taken over the $\mathrm{U}(N)$ group with the Haar measure normalized by the volume factor $\mathrm{Vol}(\mathrm{U}(N))$, and $\Delta(x)$ is the Vandermonde determinant. Thus we have the eigenvalue representation of (20):

$$\int d^N x\, \frac{\Delta(x)}{\Delta(a)} \prod_{j=1}^{N} e^{-\frac{1}{g_s} W(x_j) + a_j x_j} \prod_{j=1}^{N} \prod_{\alpha=1}^{M} (\lambda_\alpha - x_j). \tag{23}$$

We here apply the formula

$$\Delta(x)\Delta(\lambda) \prod_{\alpha=1}^{M} \prod_{j=1}^{N} (\lambda_\alpha - x_j) = \Delta(x, \lambda). \tag{24}$$

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The RHS of this equation $\Delta(x, \lambda)$ is the Vandermonde determinant for $N + M$ variables $(x_1, \ldots, x_N, \lambda_1, \ldots, \lambda_M)$, which is represented as an $(N + M) \times (N + M)$ determinant:

$$\Delta(x, \lambda) = \det \begin{pmatrix} x_k^{j-1} & x_k^{N+\alpha-1} \\ \lambda_\beta^{j-1} & \lambda_\beta^{N+\alpha-1} \end{pmatrix} = (-1)^{MN} \det \begin{pmatrix} \lambda_\beta^{\alpha-1} & \lambda_\beta^{M+j-1} \\ x_k^{\alpha-1} & x_k^{M+j-1} \end{pmatrix}. \tag{25}$$

We recall that $j, k = 1, \ldots, N$ and $\alpha, \beta = 1, \ldots, M$. Since each matrix element can be replaced with any monic polynomials, $P_k(x) = x^k + \cdots$, this determinant (25) is written in a more general form:

$$\Delta(x, \lambda) = \det \begin{pmatrix} P_{j-1}(x_k) & P_{N+\alpha-1}(x_k) \\ P_{j-1}(\lambda_\beta) & P_{N+\alpha-1}(\lambda_\beta) \end{pmatrix} = (-1)^{MN} \det \begin{pmatrix} P_{\alpha-1}(\lambda_\beta) & P_{M+j-1}(\lambda_\beta) \\ P_{\alpha-1}(x_k) & P_{M+j-1}(x_k) \end{pmatrix}. \tag{26}$$

Thus the integral is given by

$$\frac{1}{\Delta(a)\Delta(\lambda)} \int d^N x \, \Delta(x, \lambda) \prod_{j=1}^{N} e^{-\frac{1}{g_s} W(x_j) + a_j x_j}. \tag{27}$$

To perform this integral, we then introduce an auxiliary function:

$$Q_k(a) = \int dx \, P_k(x) \, e^{-\frac{1}{g_s} W(x) + ax}. \tag{28}$$

Using this function, we arrive at the final expression of the partition function (20):

$$\begin{aligned} \Psi_{N,M}\left(\{a_j\}_{j=1}^N; \{\lambda_\alpha\}_{\alpha=1}^M\right) &= \frac{1}{\Delta(a)\Delta(\lambda)} \det \begin{pmatrix} Q_{j-1}(a_k) & Q_{N+\alpha-1}(a_k) \\ P_{j-1}(\lambda_\beta) & P_{N+\alpha-1}(\lambda_\beta) \end{pmatrix} \\ &= \frac{(-1)^{MN}}{\Delta(a)\Delta(\lambda)} \det \begin{pmatrix} P_{\alpha-1}(\lambda_\beta) & P_{M+j-1}(\lambda_\beta) \\ Q_{\alpha-1}(a_k) & Q_{M+j-1}(a_k) \end{pmatrix}. \end{aligned} \tag{29}$$

This expression is manifestly symmetric under the exchange of $(a_1, \ldots, a_N)$ and $(\lambda_1, \ldots, \lambda_M)$ with the transformation $P_k(\lambda) \leftrightarrow Q_k(a)$. As seen in (28), this is nothing but a Fourier (Laplace) transform of $x^k e^{-\frac{1}{g_s} W(x)}$. Thus we can say that the characteristic polynomial and the external source in the matrix model are dual to each other in the sense of Fourier transformation.

In terms of the topological strings, this duality reflects the symplectic invariance of the canonical pair (p, x) in the B-model, as seen in (6). Thus such a symmetry, which allows the exchange of the canonical variables x and p, is directly related to the duality for the (r, s) minimal model described by $H(p, x) = p^r + x^s + \cdots$, and also the open/closed string duality [9–11]. It should also be mentioned that the symplectic invariance, or $SL(2, \mathbb{Z})$ symmetry exchanging the canonical pair, realizes the S-duality of the topological string or topological M-theory [12]. Since the two descriptions of the non-compact branes are just converted into each other through the Fourier transformation, they are essentially equivalent. Although this equivalence has already been pointed out for the Gaussian matrix model, as in Ref. [2], we can see that this symmetry even holds with a generic matrix potential $W(x)$. This matrix potential determines the geometry of the Calabi–Yau threefold in the form of (3). In addition, the positions of the branes are in a relationship satisfying the uncertainty principle: one cannot determine both the x and p coordinates at the same time, but only one or the other. We also note that this kind of symplectic invariance appears quite generally in the topological expansion of the spectral curve [13–15].

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3.2. Integrability

The formula (29) is a quite natural generalization of the well known formulae for the expectation value of characteristic polynomial product

$$\left\langle \prod_{\alpha=1}^{M} \det(\lambda_\alpha - X) \right\rangle = \frac{1}{\Delta(\lambda)} \det_{1\le\alpha,\beta\le M} P_{N+\alpha-1}(\lambda_\beta), \tag{30}$$

where $P_k(x)$ is the kth monic orthogonal polynomial with respect to the weight function $w(x) = e^{-\frac{1}{g_s}W(x)}$, and also the matrix integral with the external source

$$\int dX e^{-\frac{1}{g_s}\mathrm{Tr}\, W(X)+\mathrm{Tr}\, AX} = \frac{1}{\Delta(a)} \det_{1\le j,k\le N} Q_{j-1}(a_k). \tag{31}$$

It is convenient to apply the simplest choice of the polynomial $P_k(x) = x^k$ to this formula (31). In this case the function $Q_k(a)$ is given by

$$Q_k(a) = \int dx\, x^k\, e^{-\frac{1}{g_s}W(x)+ax} = \left(\frac{d}{da}\right)^k Q(a), \tag{32}$$

with an Airy-like function

$$Q(a) = \int dx\, e^{-\frac{1}{g_s}W(x)+ax}; \tag{33}$$

see Ref. [3] and references therein for details.

It is known that this kind of determinantal formula generically plays a role as the τ-function [6], and satisfies the Toda lattice equation by taking the equal parameter limit; see, e.g., Ref. [16]. We show that the formula (29) indeed satisfies a similar integrable equation in the following.

Let us parametrize the positions of branes by “center of mass” and deviations from it as $a_j = a + \delta a_j$ and $\lambda_\alpha = \lambda + \delta\lambda_\alpha$. We rewrite the numerator in terms of the deviations $\{\delta a_j\}$ and $\{\delta\lambda_\alpha\}$ by considering the Taylor expansion around the centers of mass:

$$\det\begin{pmatrix} Q_{j-1}(a_k) & Q_{N+\alpha-1}(a_k) \\ P_{j-1}(\lambda_\beta) & P_{N+\alpha-1}(\lambda_\beta) \end{pmatrix} = \det\begin{pmatrix} \frac{(\delta a_k)^{l-1}}{(l-1)!} & \\ & \frac{(\delta\lambda_\beta)^{\gamma-1}}{(\gamma-1)!} \end{pmatrix} \det\begin{pmatrix} Q^{(l-1)}_{j-1}(a) & Q^{(l-1)}_{N+\alpha-1}(a) \\ P^{(\gamma-1)}_{j-1}(\lambda) & P^{(\gamma-1)}_{N+\alpha-1}(\lambda) \end{pmatrix}, \tag{34}$$

where $P^{(\gamma-1)}_{j-1}(\lambda) = \left(\frac{d}{d\lambda}\right)^{\gamma-1} P_{j-1}(\lambda)$, $Q^{(l-1)}_{j-1}(a) = \left(\frac{d}{da}\right)^{l-1} Q_{j-1}(a) = Q_{j+l-2}(a)$ and so on. The first determinant in the RHS is almost canceled by the Vandermonde determinants in the denominator of (29), since they are invariant under the constant shift as $\Delta(a) = \Delta(\delta a) = \det(\delta a_k)^{j-1}$ and $\Delta(\lambda) = \Delta(\delta\lambda) = \det(\delta\lambda_\beta)^{\alpha-1}$, respectively. Therefore the partition function in the equal position limit becomes

$$\Psi_{N,M}\left(\{a_j = a\}_{j=1}^N; \{\lambda_\alpha = \lambda\}_{\alpha=1}^M\right) = c_{N,M} \det\begin{pmatrix} Q^{(k-1)}_{j-1}(a) & Q^{(k-1)}_{N+\alpha-1}(a) \\ P^{(\beta-1)}_{j-1}(\lambda) & P^{(\beta-1)}_{N+\alpha-1}(\lambda) \end{pmatrix} \tag{35}$$

$$= (-1)^{MN} c_{N,M} \det\begin{pmatrix} P^{(\beta-1)}_{\alpha-1}(\lambda) & P^{(\beta-1)}_{M+j-1}(\lambda) \\ Q^{(k-1)}_{\alpha-1}(a) & Q^{(k-1)}_{M+j-1}(a) \end{pmatrix} \tag{36}$$

$$\equiv c_{N,M} \det R_{N,M}(a,\lambda),$$

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with

$$c_{N,M} = \prod_{j=1}^{N} \frac{1}{(j-1)!} \prod_{\alpha=1}^{M} \frac{1}{(\alpha-1)!}, \tag{37}$$

which corresponds to the volume factor for the U(N) and U(M) groups. This expression is seen as a hybridized version of the Wronskian. In the following we apply $P_k(x) = x^k$ and (32) as in the case of (31) for simplicity.

In order to derive the integrable equation, we now use the Jacobi identity for determinants, which is given by

$$D \cdot D \begin{pmatrix} i & j \\ k & l \end{pmatrix} = D \begin{pmatrix} i \\ k \end{pmatrix} \cdot D \begin{pmatrix} j \\ l \end{pmatrix} - D \begin{pmatrix} i \\ l \end{pmatrix} \cdot D \begin{pmatrix} j \\ k \end{pmatrix}, \tag{38}$$

where D is a determinant, and the minor determinant $D\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)$ is obtained by removing the ith row and the jth column from the matrix. Similarly. $D\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right)$ is obtained by eliminating the i, jth row and the k, lth column. Putting $i = k = N + M$ and $j = l = N + M - 1$ for the determinant in (36), we have

$$\begin{aligned} \det R_{N,M} \cdot \det R_{N-2,M} = {} & \det R_{N-1,M} \cdot M\lambda\partial_a^2 \det R_{N-1,M} \\ & - \partial_a \det R_{N-1,M} \cdot M\lambda\partial_a \det R_{N-1,M}. \end{aligned} \tag{39}$$

Here we have used a formula (A1) discussed in Appendix A. This provides the following relation for the equal position partition function:

$$\frac{\Psi_{N+1,M} \cdot \Psi_{N-1,M}}{\left(\Psi_{N,M}\right)^2} = \frac{M}{N}\lambda \frac{\partial^2}{\partial a^2} \log \Psi_{N,M}. \tag{40}$$

This is just the Toda lattice equation along the a-direction, but with a trivial factor that can be removed by rescaling the function.

We can assign another relation to the partition function by the identity (38) for the expression (35) with $i = k = N + M$ and $j = l = N + M - 1$, which reads

$$\begin{aligned} \det R_{N,M} \cdot \det R_{N,M-2} = {} & \det R_{N,M-1} \cdot (M-1)\partial_a\partial_\lambda\lambda \det R_{N,M-1} \\ & - \partial_\lambda \det R_{N,M-1} \cdot (M-1)\lambda\partial_a \det R_{N,M-1}. \end{aligned} \tag{41}$$

We have again used the relation (A1). Rewriting this relation in terms of the partition function (35), we have

$$\frac{\Psi_{N,M+1} \cdot \Psi_{N,M-1}}{\left(\Psi_{N,M}\right)^2} = \frac{\partial^2}{\partial a \partial \lambda} \lambda \log \Psi_{N,M}. \tag{42}$$

We then obtain the 2D Toda lattice equation [17] with an extra factor. In order to remove this irrelevant factor, we rescale the partition function:

$$\tilde{\Psi}_{N,M}(a,\lambda) = e^{-\lambda}\, \Psi_{N,M}(a,\lambda). \tag{43}$$

Thus the Toda lattice equations (40) and (42) are now written in the well known form[1]

$$\frac{\tilde{\Psi}_{N+1,M} \cdot \tilde{\Psi}_{N-1,M}}{\left(\tilde{\Psi}_{N,M}\right)^2} = \frac{M}{N} \frac{\partial^2}{\partial a^2} \log \tilde{\Psi}_{N,M}, \qquad \frac{\tilde{\Psi}_{N,M+1} \cdot \tilde{\Psi}_{N,M-1}}{\left(\tilde{\Psi}_{N,M}\right)^2} = \frac{\partial^2}{\partial a \partial \lambda} \log \tilde{\Psi}_{N,M}. \tag{44}$$

[1] For example, see Ref. [18] for the bilinear form of the Toda lattice equation with the τ-functions.

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This means that the brane partition function (43) plays the role of the τ-function for the 1D and 2D Toda lattice equations simultaneously. We note that this is an exact result for finite N (and also M). If one takes the large N limit, corresponding to the continuum limit for the Toda lattice equations, it reduces to the Korteweg-de Vries/Kadomtsev-Petviashvili (KdV/KP) equations. We also comment that the τ-function of the 2D Toda lattice hierarchy can be realized as the two-matrix model integral [19,20].

Although in this section we have focused only on the equal position limit of the partition function (29), it can be regarded as the τ-function for the corresponding integrable hierarchy. In this case we can introduce two kinds of Miwa coordinates:

$$t_n = \frac{1}{n}\,\mathrm{Tr}\,A^{-n}, \quad \tilde{t}_n = \frac{1}{n}\,\mathrm{tr}\,\Lambda^{-n}. \tag{45}$$

It is shown that all the time variables, t_n and $\tilde{t}_n$, are trivially related to each other in the equal parameter limit. After taking the continuum limit, namely the large N limit of the matrix model, it shall behave as the τ-function for the KdV/KP hierarchies.

4. Gaussian matrix model

We now study a specific example of the matrix model with the harmonic potential $W(x) = \frac{1}{2}x^2$, namely the Gaussian matrix model. In this case we can check the duality formula more explicitly [21–23]:

$$\frac{1}{\mathcal{Z}_N}\int dX\, e^{-\frac{1}{2g_s}\mathrm{Tr}(X-A)^2}\prod_{\alpha=1}^{M}\det(\lambda_\alpha - X) = (-1)^{MN}\frac{1}{\mathcal{Z}_M}\int dY\, e^{-\frac{1}{2g_s}\mathrm{tr}(Y-i\Lambda)^2}\prod_{j=1}^{N}\det(a_j + iY), \tag{46}$$

where X and Y are Hermitian matrices with matrix sizes $N\times N$ and $M\times M$, and “Tr” and “tr” stand for the trace for them, respectively. This equality is just rephrased as

$$e^{-\frac{1}{2g_s}\mathrm{Tr}\,A^2}\left\langle e^{\frac{1}{g_s}\mathrm{Tr}\,XA}\prod_{\alpha=1}^{M}\det(\lambda_\alpha - X)\right\rangle = (-1)^{MN}e^{\frac{1}{2g_s}\mathrm{tr}\,\Lambda^2}\left\langle e^{\frac{1}{g_s}i\,\mathrm{tr}\,Y\Lambda}\prod_{j=1}^{N}\det(a_j + iY)\right\rangle. \tag{47}$$

From the generic formula (29), we can understand this duality as a consequence of the self-duality of Hermite polynomials with respect to Fourier transformation. Actually, the Hermite polynomial, which is an orthogonal polynomial with the Gaussian weight function $w(x) = e^{-\frac{1}{2}x^2}$, has an integral representation:

$$H_k(x) = \int_{-\infty}^{\infty}\frac{dt}{\sqrt{2\pi}}\,(it)^k\, e^{-\frac{1}{2}(t+ix)^2}. \tag{48}$$

This expression is essentially the same as (32) in the case of the harmonic potential. This is a specific property for the Gaussian model.

4.1. *Fermionic formula*

For the Gaussian matrix model, there is another interesting derivation of the duality (46) using fermionic variables, instead of the method used in Sect. 3.1. Following the approach applied in Refs. [21,22], basically, we discuss it from the viewpoint of the topological strings. We will actually show that the effective fermionic action for this partition function gives a quite natural perspective on the topological brane.

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Introducing fermionic variables in bifundamental representations $(N, \bar{M})$ and $(\bar{N}, M)$ of $\mathrm{U}(N) \times \mathrm{U}(M)$, the characteristic polynomial is involved in an exponential form:

$$\prod_{\alpha=1}^{M} \det(\lambda_\alpha - X) = \int d[\bar{\psi}, \psi]\, e^{\sum_{\alpha=1}^{M} \bar{\psi}_i^\alpha (\lambda_\alpha \ -X)_{ij} \psi_j^\alpha}. \tag{49}$$

Using this formula, the LHS of (47) becomes

$$\frac{1}{\mathcal{Z}_N} \int d[X, \bar{\psi}, \psi]\, e^{-\frac{1}{2g_s} \mathrm{Tr}(X-A)^2 + \bar{\psi}_i^\alpha (\lambda_\alpha \ -M)_{ij} \psi_j^\alpha}. \tag{50}$$

Then the effective action yields

$$\begin{aligned} S_{\text{eff}}(X, \bar{\psi}, \psi) &= -\frac{1}{2g_s} \mathrm{Tr}(X - A)^2 + \bar{\psi}_i^\alpha (\lambda_\alpha \mathbb{1} - M)_{ij} \psi_j^\alpha \\ &= -\frac{1}{2g_s} \mathrm{Tr}(X - A - g_s \psi^\alpha \bar{\psi}^\alpha)^2 + \mathrm{Tr}\, A\, \psi^\alpha \bar{\psi}^\alpha + \frac{g_s}{2} \psi_i^\alpha \bar{\psi}_j^\alpha \psi_j^\beta \bar{\psi}_i^\beta - \mathrm{tr}\, \Lambda\, \psi_j \bar{\psi}_j, \end{aligned} \tag{51}$$

where $(\psi^\alpha \bar{\psi}^\alpha)_{ij} \equiv \psi_i^\alpha \bar{\psi}_j^\alpha$ and $(\psi_j \bar{\psi}_j)^{\alpha\beta} \equiv \psi_j^\alpha \bar{\psi}_j^\beta$ are $N \times N$ and $M \times M$ matrices, respectively. Integrating out the matrix X, we obtain the intermediate form of the formula, which can be represented only in terms of the fermionic variables

$$e^{-\frac{1}{2g_s} \mathrm{Tr}\, A^2} \left\langle e^{\frac{1}{g_s} \mathrm{Tr}\, X A} \prod_{\alpha=1}^{M} \det(\lambda_\alpha - X) \right\rangle = \int d[\bar{\psi}, \psi]\, e^{\frac{g_s}{2} \psi_i^\alpha \bar{\psi}_j^\alpha \psi_j^\beta \bar{\psi}_i^\beta + \mathrm{Tr}\, A\, \psi^\alpha \bar{\psi}^\alpha - \mathrm{tr}\, \Lambda\, \psi_j \bar{\psi}_j}. \tag{52}$$

Since the four-point interaction is also represented in terms of the $M \times M$ matrix as $\mathrm{Tr}\left(\psi^\alpha \bar{\psi}^\alpha\right)^2 = -\,\mathrm{tr}\left(\psi_i \bar{\psi}_i\right)^2$, this term can be removed by inserting an $M \times M$ auxiliary matrix Y:

$$\begin{aligned} &\frac{1}{\mathcal{Z}_M} \int d[Y, \bar{\psi}, \psi]\, e^{-\frac{1}{2g_s} \mathrm{tr}(Y - i\Lambda - i g_s \psi_j \bar{\psi}_j)^2 - \mathrm{tr}\, \Lambda\, \psi_j \bar{\psi}_j - \frac{g_s}{2} \mathrm{tr}\left(\psi_i \bar{\psi}_i\right)^2 + \mathrm{Tr}\, A\, \psi^\alpha \bar{\psi}^\alpha} \\ &= \frac{(-1)^{MN}}{\mathcal{Z}_M} \int dY \prod_{j=1}^{N} \det(a_j + iY)\, e^{-\frac{1}{2g_s} \mathrm{tr}(Y - i\Lambda)^2}. \end{aligned} \tag{53}$$

This is just the RHS of the duality formula (47).

Let us comment on the meaning of this formula in terms of the topological strings. When we apply m distinct values to A as

$$A = \mathrm{diag}(\underbrace{a^{(1)}, \ldots, a^{(1)}}_{N_1}, \underbrace{a^{(2)}, \ldots, a^{(2)}}_{N_2}, \ldots\ldots, \underbrace{a^{(m)}, \ldots, a^{(m)}}_{N_m}), \tag{54}$$

stacked N branes are decoupled into $N_1 + \cdots + N_m$, as shown in Fig. 2. This means that the $\mathrm{U}(N)$ symmetry of the original matrix model is broken into its subsector:

$$\mathrm{U}(N) \longrightarrow \mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_m). \tag{55}$$

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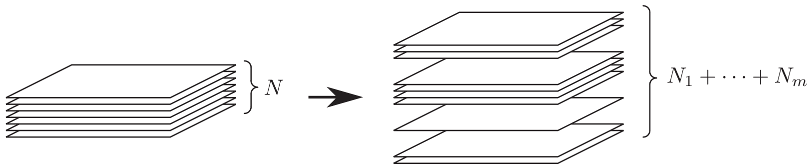

Fig. 2. When we turn on the external source A, the degenerate positions of N branes are lifted. Correspondingly, the $\mathrm{U}(N)$ symmetry is broken into $\mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_m)$ by applying m distinct values to A.

We find a similar symmetry breaking in the dual representation. In particular, when we put Λ as

$$\Lambda = \mathrm{diag}(\underbrace{\lambda^{(1)}, \ldots, \lambda^{(1)}}_{M_1}, \underbrace{\lambda^{(2)}, \ldots, \lambda^{(2)}}_{M_2}, \ldots\ldots, \underbrace{\lambda^{(l)}, \ldots, \lambda^{(l)}}_{M_l}), \tag{56}$$

the $\mathrm{U}(M)$ symmetry is broken as

$$\mathrm{U}(M) \longrightarrow \mathrm{U}(M_1) \times \cdots \times \mathrm{U}(M_l). \tag{57}$$

We can also discuss the symmetry breaking of the fermions by seeing the fermionic effective action in (52),

$$S_{\text{eff}}(\psi, \bar{\psi}) = \frac{g_s}{2} \psi_i^\alpha \bar{\psi}_j^\alpha \psi_j^\beta \bar{\psi}_i^\beta + \mathrm{Tr}\, A\, \psi^\alpha \bar{\psi}^\alpha - \mathrm{tr}\, \Lambda\, \psi_j \bar{\psi}_j. \tag{58}$$

Although the four-point interaction term is invariant under the full symmetry of $\mathrm{U}(N) \times \mathrm{U}(M)$, this symmetry is partially broken, because the source term plays the role of the non-singlet mass term. The remaining symmetry is $\mathrm{U}(N_i) \times \mathrm{U}(M_j)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, l$, as a subset of $\mathrm{U}(N) \times \mathrm{U}(M)$. This fermionic excitation should be seen as a remnant of the chiral fermion associated with the anomaly on the intersecting branes [24–28].

4.2. *Bosonic formula*

We can extend the duality formula (47) for the inverse characteristic polynomial [23]:

$$e^{-\frac{1}{2g_s}\mathrm{Tr}\, A^2} \left\langle e^{\frac{1}{g_s}\mathrm{Tr}\, XA} \prod_{\alpha=1}^{M} \det(\lambda_\alpha - X)^{-1} \right\rangle = e^{-\frac{1}{2g_s}\mathrm{tr}\, \Lambda^2} \left\langle e^{\frac{1}{g_s}\mathrm{tr}\, Y\Lambda} \prod_{j=1}^{N} \det(a_j - Y)^{-1} \right\rangle. \tag{59}$$

In this case a bifundamental bosonic field plays a similar role to the fermionic field, which is used to represent the characteristic polynomial in the numerator. Actually, we can derive this duality formula in almost the same manner as that discussed in Sect. 4.1.

The average shown in the LHS of (59) is explicitly written as

$$e^{-\frac{1}{2g_s}\mathrm{Tr}\, A^2} \left\langle e^{\frac{1}{g_s}\mathrm{Tr}\, XA} \prod_{\alpha=1}^{M} \det(\lambda_\alpha - X)^{-1} \right\rangle = \frac{1}{\mathcal{Z}_N} \int dX \prod_{\alpha=1}^{M} \det(\lambda_\alpha - X)^{-1} e^{-\frac{1}{2g_s}\mathrm{Tr}(X-A)^2}. \tag{60}$$

Since the inverse of a determinant is written as a Gaussian integral with a bosonic variable in the bifundamental representations,

$$\prod_{\alpha=1}^{M} \det(\lambda_\alpha - X)^{-1} = \int d[\phi, \phi^\dagger]\, e^{-\sum_{\alpha=1}^{M} \phi_i^{*\alpha} (\lambda_\alpha \ -X)_{ij} \phi_j^\alpha}, \tag{61}$$

the intermediate form, corresponding to (52), is given by

$$\int d[\phi, \phi^\dagger]\, e^{\frac{g_s}{2}\phi_i^\alpha \phi_j^{*\alpha} \phi_j^\beta \phi_i^{*\beta} + \mathrm{Tr}\, A\, \phi^\alpha \phi^{*\alpha} - \mathrm{tr}\, \Lambda\, \phi_j \phi_j^*}. \tag{62}$$

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Then, inserting an auxiliary $M \times M$ Hermitian matrix Y in order to eliminate the four-point interaction, it is written as

$$\frac{1}{\mathcal{Z}_M} \int d[Y, \phi, \phi^\dagger]\, e^{-\frac{1}{2g_s} \operatorname{tr}(Y - \Lambda + g_s \phi_j \phi_j^*)^2 - \operatorname{tr} \Lambda\, \phi_j \phi_j^* + \frac{g_s}{2} \operatorname{tr}(\phi_i \phi_i^*)^2 + \operatorname{Tr} A\, \phi^\alpha \phi^{*\alpha}}$$
$$= e^{-\frac{1}{2g_s} \operatorname{tr} \Lambda^2} \left\langle e^{\frac{1}{g_s} \operatorname{tr} Y \Lambda} \prod_{j=1}^{N} \det(a_j - Y)^{-1} \right\rangle. \tag{63}$$

This is the RHS of the duality formula (59).

The symmetry of the bifundamental bosons is partially broken, as seen in the bosonic effective action (62), as well as the fermionic case. The difference from the previous case is the role of branes. From the topological string point of view, the inverse of the characteristic polynomial is regarded as the partition function of the non-compact anti-brane, which is created by the vertex operator with the opposite charge $\overline{V}(x) = e^{\frac{1}{2g_s}\phi(x)}$, instead of (12). In this case, the correlation function is written in terms of the Cauchy transform of the corresponding orthogonal polynomial [29].

5. Discussion

In this article we have investigated the symplectic invariance of branes in the topological B-model using its matrix model description. In particular, since two different descriptions of the non-compact brane correspond to the characteristic polynomial and the external source in the matrix model, we have considered the brane partition function given by inserting both of them simultaneously. We have derived the determinantal formula for this partition function, and shown that the two descriptions of the branes are dual to each other in the sense of the Fourier transformation. We have also shown that the brane partition function plays the role of the τ-function, and satisfies the Toda lattice equations in one and two dimensions. We have investigated the Gaussian matrix model as an example, and discussed the effective action of the topological branes in terms of the bifundamental fermion/boson.

Although we have focused on the $\mathrm{U}(N)$ symmetric matrix model throughout this article, we can apply essentially the same argument to $\mathrm{O}(N)$ and $\mathrm{Sp}(2N)$ symmetric matrix models. In such a case the Hermitian matrix is replaced with real symmetric and self-dual quaternion matrices, respectively. Actually, when the Gaussian potential is assigned, one can obtain a similar duality formula [23,30,31], which claims that the dual of the $\mathrm{O}(N)$ model is the $\mathrm{Sp}(2N)$ model and vice versa. This relation is extended to an arbitrary β-ensemble, and a generic duality between β and $1/\beta$ is found. From the string theoretical point of view, this property is naturally understood as insertion of an orientifold plane. Correspondingly, the $\mathrm{U}(N) \times \mathrm{U}(M)$ bifundamental fermion/boson used in Sect. 4 is replaced with the $\mathrm{O}(N) \times \mathrm{Sp}(2M)$ bifundamental variables.

Let us comment on some possible applications of the duality discussed in this article. The exact low energy dynamics of $\mathcal{N} = 2$ gauge theory, which is described by Seiberg–Witten theory, is solved by the world volume theory of D4-branes suspended between NS5-branes, and also an M5-brane, appearing in its M-theory lift [32]. In this case, since the geometry of this M5-brane indicates the Seiberg–Witten curve of the corresponding $\mathcal{N} = 2$ theory, the positions of branes are directly related to the gauge theory dynamics. Thus it is expected that a nontrivial gauge theory duality is derived from the duality between the two coordinates of branes. Actually, a similar duality is discussed along this direction [33].

From the matrix model perspective, it is interesting to consider the ratio of the characteristic polynomials in the presence of the external source [29], and its interpretation in terms of topological

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strings. The characteristic polynomial in the numerator and the denominator plays the role of the creation operator for the brane and anti-brane. Thus the ratio should describe the pair creation and annihilation of branes. In particular, it is expected that the scaling limit of the ratio extracts some interesting features of the tachyon condensation in topological strings. This kind of problem is also interesting in the context of the matrix model itself, because one can often find an universal property of the matrix model in such a scaling limit.

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Appendix A. A formula for determinants

In order to obtain (39) and (41), it is convenient to use the following relation:

$$\frac{\det B_M(x)}{\det A_M(x)} = M\,x, \tag{A1}$$

where we have introduced two $M \times M$ matrices:

$$A_M = \begin{pmatrix} \left(x^N\right)^{(0)} & \cdots & \left(x^{N+M-2}\right)^{(0)} & \left(x^{N+M-1}\right)^{(0)} \\ \left(x^N\right)^{(1)} & \cdots & \left(x^{N+M-2}\right)^{(1)} & \left(x^{N+M-1}\right)^{(1)} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ \left(x^N\right)^{(M-1)} & \cdots & \left(x^{N+M-2}\right)^{(M-1)} & \left(x^{N+M-1}\right)^{(M-1)} \end{pmatrix}, \tag{A2}$$

and

$$B_M = \begin{pmatrix} \left(x^N\right)^{(0)} & \cdots & \left(x^{N+M-2}\right)^{(0)} & \left(x^{N+M}\right)^{(0)} \\ \left(x^N\right)^{(1)} & \cdots & \left(x^{N+M-2}\right)^{(1)} & \left(x^{N+M}\right)^{(1)} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ \left(x^N\right)^{(M-1)} & \cdots & \left(x^{N+M-2}\right)^{(M-1)} & \left(x^{N+M}\right)^{(M-1)} \end{pmatrix}. \tag{A3}$$

Here we denote $(x^j)^{(l)} = (d/dx)^l x^j$ and so on. Using the Jacobi identity (38) for $\det A_M$ with $i = k = M$ and $j = l = M - 1$, we have

$$\det A_M \cdot \det A_{M-2} = \det A_{M-1} \cdot \partial_x \det B_{M-1} - \partial_x \det A_{M-1} \cdot \det B_{M-1}. \tag{A4}$$

It is convenient to rewrite this relation as

$$\frac{\det A_M \cdot \det A_{M-2}}{(\det A_{M-1})^2} = \frac{\partial}{\partial x}\left(\frac{\det B_{M-1}}{\det A_{M-1}}\right). \tag{A5}$$

This is interpreted as a remnant of the Toda lattice equation [6].

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What we have to do next is to evaluate det A_M. To obtain this, we consider a ratio of determinants, and then take the equal parameter limit, $x_\alpha \to x$ for all $\alpha = 1, \ldots, M$,

$$\frac{1}{\Delta(x)} \det_{1\le\alpha,\beta\le M} \left(x_\beta\right)^{N+\alpha-1} = \prod_{\alpha=1}^{M} x_\alpha^N \xrightarrow{x_\alpha\to x} x^{MN}. \tag{A6}$$

On the other hand, it is written in another form:

$$\frac{1}{\Delta(x)} \det_{1\le\alpha,\beta\le M} \left(x_\beta\right)^{N+\alpha-1} \xrightarrow{x_\alpha\to x} \left(\prod_{\alpha=1}^{M}(\alpha-1)!\right)^{-1} \det_{1\le\alpha,\beta\le M} \left(x^{N+\alpha-1}\right)^{(\beta-1)}. \tag{A7}$$

Comparing these two expressions, we obtain the following result:

$$\det A_M = \left(\prod_{\alpha=1}^{M}(\alpha-1)!\right) x^{MN}. \tag{A8}$$

Substituting this expression into the relation (A5), we arrive at the relation (A1).

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